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Near operators theory and fully nonlinear elliptic equations

Antonio Tarsia

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Abstract We give a short survey of the Campanato *near operators theory* and of its applications to fully nonlinear elliptic equations.

Keywords Fully nonlinear elliptic equations · Near operators theory

1 Introduction

One of the last contributions of Sergio Campanato to Mathematical Analysis is what is now known as the theory of *near operators*. This lecture deals with this theory and, particularly, on its most recent developments and applications.

We begin by giving the definition of *nearness* between operators.

Definition 1 Let $\mathcal{X}$ be a set, $\mathcal{B}$ be a Banach space with norm $\|\cdot\|_{\mathcal{B}}$, A and B be two operators such that $A,\ B : \mathcal{X} \to \mathcal{B}$. We say that A is near B if there are two positive constants, α, k, with $0 < k < 1$, such for every $x_1,\ x_2 \in \mathcal{X}$ we have:

$$\|B(x_1) - B(x_2) - \alpha[A(x_1) - A(x_2)]\|_{\mathcal{B}} \leq k\ \|B(x_1) - B(x_2)\|_{\mathcal{B}}. \tag{1}$$

The starting point of the theory is the following theorem (see references on Sect. 3).

Theorem 1 *Let $\mathcal{X}$ be a set, $\mathcal{B}$ be a Banach space with norm $\|\cdot\|_{\mathcal{B}}$, A and B be two operators such that $A,\ B : \mathcal{X} \to \mathcal{B}$, and let A be near B. Under these hypotheses, if B is a bijection between $\mathcal{X}$ and $\mathcal{B}$, A is also a bijection between $\mathcal{X}$ and $\mathcal{B}$.*

This theorem allows us to show existence and uniqueness of the solution for a class of fully nonlinear equations as the following:

$$\begin{cases} u \in H^{2,2} \cap H_0^{1,2}(\Omega) \\ F(x, D^2u) = f(x), \quad \text{a.e. in } \Omega. \end{cases} \tag{2}$$

A. Tarsia (✉)
Dipartimento di Matematica "L. Tonelli", Università degli Studi di Pisa, Largo B. Pontecorvo, 5., Pisa I-56127, Italy
e-mail: tarsia@dm.unipi.it

where $f \in L^2(\Omega)$, $D^2u = \{D_{ij}u\}_{i,j=1,\ldots,n}$, Ω is an open bounded set of $\mathbf{R}^n$, $F : \Omega\times\mathcal{S}^n \to \mathbf{R}$ is measurable in x, continuous in the other variables, $\mathcal{S}^n$ denotes the $\frac{n(n+1)}{2}$-dimensional space of real symmetric $n \times n$ matrices.

To solve this problem, we assume here the following hypotheses, that we still call *Condition A* even if it is a generalization of usual Campanato's *Condition A* (see [3]):

Definition 2 (*Condition A*) There exist three positive constants a, γ and δ with $\gamma + \delta < 1$, $\delta \geq 0$, such that for all $M,\ N \in \mathcal{S}^n$ we have

$$\left|\sum_{i=1}^{n} n_{ii} - a[F(x, M + N) - F(x, M)]\right| \leq \gamma\|N\| + \delta \left|\sum_{i=1}^{n} n_{ii}\right|, \tag{3}$$

where $N = \{n_{ij}\}_{i,j=1,\ldots,n}$, and $\|N\|^2 = \sum_{i,j=1}^{n} n_{ij}^2$.

Theorem 2 *If F verifies Condition A, $F(x, 0) \in L^2(\Omega)$ and Ω is convex then the Problem (2) is well posed.*

Proof We set

$$Bu = \Delta u, \quad A(u) = F(x, D^2u) \tag{4}$$

$$\mathcal{X} = H^{2,2} \cap H_0^{1,2}(\Omega), \quad \mathcal{B} = L^2(\Omega) \tag{5}$$

We assume $F(x, 0) = 0$, a.e. in Ω (otherwise we consider $\mathcal{F}(x, M) = F(x, M)-F(x, 0)$) and observe that $A(u) \in L^2(\Omega)$. Indeed, by *Condition A* we can write

$$\begin{aligned}\int_\Omega |a\ F(x, D^2 u(x))|^2\ dx &\leq 2 \int_\Omega \left\{|\Delta u(x)\ -\ a\,F(x, D^2u(x))|^2\ +\ |\Delta u(x)|^2\right\}\ dx\\ &\leq 2\,[(\gamma\ +\ \delta)^2 + 1] \int_\Omega |\Delta u(x)|^2\ dx.\end{aligned}$$

Moreover the operators defined by (4), under *Condition A* verify Definition 1, namely A is near B in the spaces defined by (5). Indeed

$$\begin{aligned}\|Bu - Bv{-}\alpha[A(u){-}A(v)]\|_{\mathcal{B}}^2 &= \int_\Omega \left|\Delta(u - v) - \alpha[F(x, D^2u) - F(x, D^2v)]\right|^2\ dx\\ &\leq \int_\Omega \left(\gamma\,\|D^2(u - v)\|\ +\ \delta\,|\Delta(u - v)|\right)^2\ dx\\ &\leq\ \gamma(\gamma + \delta) \int_\Omega \|D^2(u - v)\|^2\,dx\ +\ \delta(\gamma + \delta)\\ &\quad\times \int_\Omega |\Delta(u - v)|^2\,dx\ \text{(because}\ \ \Omega\ \text{ is convex)}\\ &\leq (\gamma + \delta)^2 \int_\Omega |\Delta(u - v)|^2\ dx = (\gamma + \delta)^2\ \|B(u - v)\|_{\mathcal{B}}^2\end{aligned} \tag{6}$$

Hence (1) is fulfilled with $k = \gamma + \delta$.

As B is a bijection between $H^{2,2} \cap H_0^{1,2}(\Omega)$ and $L^2(\Omega)$, A is a bijection too (see Theorem 1). □

2 Other definitions of ellipticity

A generalization of *Condition A* is the following.

Let $F : \Omega \times \mathcal{S}^n \to \mathbf{R}$ be measurable in Ω and continuous in $\mathcal{S}^n$. We say that F verifies *Condition* A_x if:

Definition 3 (*Condition* A_x) There exist two real constants γ, δ, with $\gamma > 0, \delta \geq 0, \gamma + \delta < 1$, a positive measurable function $a : \Omega \to \mathbf{R}$ and a function $B(x) : \Omega \to \mathcal{S}^n$, with $B(x) > 0$[1] such that

$$\left|(B(x)|N) - a(x)[F(x, M+N) - F(x, M)]\right| \leq \gamma \|N\| + \delta \left|(B(x)|N)\right| \tag{7}$$

for all $M, N \in \mathcal{S}^n$, a.e. in Ω.

In the linear case, i.e. if $F(x, D^2u) = \sum_{i,j=1}^n a_{ij}(x)\, D_{ij}u = (\mathcal{A}(x)|D^2u)$, *Condition* A_x with $\delta = 0$ is equivalent to the following Condition of Cordes type:

Definition 4 (*Condition of Cordes type*) Let $\mathcal{A}(x) = \{a_{ij}(x)\}_{i,j=1,\ldots,n}$ be a family of matrices such that $\mathcal{A}(x) > 0$, a.e. in Ω and $\mathcal{A}(x) \in \mathcal{S}^n$. We say that $\mathcal{A}(x)$ satisfies the Cordes Condition if there exist $B(x) : \Omega \to \mathcal{S}^n$ and a real positive number γ such that $\|B(x)\| > \gamma$ a.e in Ω and

$$\frac{(B(x)|\mathcal{A}(x))^2}{\|\mathcal{A}(x)\|^2} \geq \|B(x)\|^2 - \gamma^2, \quad \text{a.e. in } \Omega. \tag{8}$$

Indeed, from

$$\left|(B(x)|N) - a(x)(\mathcal{A}(x)|N)\right| \leq \gamma \|N\|, \quad \forall N \in \mathcal{S}^n \tag{9}$$

the same inequality follows for all $n \times n$-matrices ξ, because $(\mathcal{A}(x)|\xi) = (\mathcal{A}(x)|\xi^{\mathrm{S}})$, where ξ^{S} is the simmetric part of ξ, i.e. $\xi^{\mathrm{S}} = \frac{\xi + {}^t\xi}{2}$. Then (9) is equivalent to

$$\|B(x) - a(x)\mathcal{A}(x)\| \leq \gamma$$

that is

$$P(a) = a(x)^2 \|\mathcal{A}(x)\|^2 - 2a(x)(B(x)|\mathcal{A}(x)) + \|B(x)\|^2 - \gamma^2 \leq 0$$

This second degree polynomial attains its minimum at

$$a_0(x) = \frac{(B(x)|\mathcal{A}(x))}{\|\mathcal{A}(x)\|^2}$$

hence $P(a_0(x)) \leq 0$ if and only if

$$\|B(x)\|^2 - \gamma^2 \leq \frac{(B(x)|\mathcal{A}(x))^2}{\|\mathcal{A}(x)\|^2}.$$

Remark 1 Suppose $n = 2, B = I$. A family of symmetric matrices $A(x) = \{a_{ij}(x)\}_{i,j=1,\ldots,n}$, with L^∞ coefficients, is uniformly elliptic on Ω if and only if it satisfies Condition of Cordes and $\|A(x)\| \geq c > 0$, $\sum_{i=1}^n a_{ii}(x) > 0$ a.e. in Ω. (See [7])

[1] we write $B > 0$ to denote a positive definite matrix, while $(B|N) = \sum_{i,j=1}^n b_{ij}n_{ij}$, where $B = \{b_{ij}\}_{i,j=1,\ldots,n}$, $N = \{n_{ij}\}_{i,j=1,\ldots,n}$, and $\|B\|^2 = \sum_{i,j=1}^n b_{ij}^2$.

Since the last decades of the 20th century fully nonlinear elliptic equations have been extensively studied. Several papers have been dedicated to these equations, assuming different definitions of ellipticity.

One of them is *Condition A* of Campanato (or *Condition* A_x) which we recalled before, another one is the following definition of uniformly elliptic operators.

We shall try to see the connection between them.

Definition 5 F is uniformly elliptic (in the sense of L.A. Caffarelli, X. Cabré, see [1], p. 12) if there are two positive real constants $\lambda \le \Lambda$ (called ellipticity constants) such that for all $x \in \Omega$ and any $M, N \in \mathcal{S}^n$, with $N \ge 0$

$$\lambda \|N\| \le F(x, M+N) - F(x, M) \le \Lambda \|N\|, \tag{10}$$

for all N (if N is a non-negative definite matrix, we write $N \ge 0$).

Definition 6 F is uniformly elliptic (in the sense of N. S. Trudinger, see [9]) if there exist a real constant μ and two positive functions λ, Λ, defined on Ω, $\lambda(x) \le \Lambda(x)$, such that for all $x \in \Omega$ and any $M, N \in \mathcal{S}^n$, with $N \ge 0$

$$\lambda(x)\ (I|N)\ \le F(x, M+N) - F(x, M) \le \Lambda(x)\ (I|N),\ \text{ and }\ \frac{\Lambda(x)}{\lambda(x)} \le \mu$$
$$\text{(where I is the identity matrix)}.$$

We show the following

Theorem 3 *If F verifies Condition* A_x *with* $0 < C_1 < a(x) < C_2$, *and* $B(x) = I$, *then F is uniformly elliptic in the sense of Definition 5.*

Proof By (7) it follows for all $M, N \in \mathcal{S}^n N \ge 0$

$$\frac{(1-\delta)(I|N) - \gamma\|N\|}{a(x)} \le F(x, M+N) - F(x, M) \le \frac{\gamma\|N\| + (1+\delta)(I|N)}{a(x)},$$

but, denoting by $\lambda_1, \ldots, \lambda_n$ the eigenvalues of N, it holds

$$(I|N) = \sum_{i=1}^{n} \lambda_i \ge \left(\sum_{i=1}^{n} \lambda_i^2\right)^{\frac{1}{2}} = \|N\|$$

and hence

$$\begin{aligned}\|N\|\ \frac{1-\delta-\gamma}{C_2} \le \|N\|\ \frac{1-\delta-\gamma}{a(x)} &\le F(x, M+N) - F(x, M)\\ &\le \frac{(1+\delta)\sqrt{n}+\gamma}{a(x)}\ \|N\| \le \frac{(1+\delta)\sqrt{n}+\gamma}{C_1}\ \|N\|.\end{aligned}$$

□

Theorem 4 *If F is uniformly elliptic, as in Definition 5, then it satisfies Condition* A_x, *for all* $B(x) : \Omega \to \mathcal{S}^n$, *such that*

$$\theta(x)\|N\| \le (B(x)|N) \le \Theta(x)\|N\|, \quad \forall N \in \mathcal{S}^n, \quad N \ge 0,$$

(where $0 < \theta(x) \le \Theta(x)$*), provided there exists* $a(x) > 0$ *such that*

$$\sup_{\Omega} \sqrt{(\Theta(x) - a(x)\lambda)^2 + (\theta(x) - a(x)\Lambda)^2} < 1.$$

To show this theorem we need the following lemma.

Lemma 1 *F is uniformly elliptic, in the sense of Definition 5 if and only if*

$$F(x, M+N) \le F(x, M) + \Lambda\|N^+\| - \lambda\|N^-\|, \quad \forall M, N \in \mathcal{S}^n, \quad \forall x \in \Omega. \tag{11}$$

This is easy to check because any matrix $N \in \mathcal{S}^n$ can be uniquely decomposed as $N = N^+ - N^-$ where $N^+, N^- \ge 0$ and $N^+N^- = 0$ (see [1], p. 12).

Proof (of Theorem 4).
By (11), we have

$$F(x, M+N) - F(x, M) \le \Lambda\|N^+\| - \lambda\|N^-\| \tag{12}$$

On the other hand by choosing in (11) $\mathcal{M} = M + N, \mathcal{N} = -NM, N \in \mathcal{S}^n$, we have

$$-\Lambda\|\mathcal{N}^-\| + \lambda\|\mathcal{N}^+\| \le F(x, \mathcal{M}+\mathcal{N}) - F(x, \mathcal{M}) \tag{13}$$

Let $B(x) : \Omega \to \mathcal{S}^n$, with $B(x) > 0$, be a matrix such that for all $N \in \mathcal{S}^n$, $N \ge 0$, we have

$$\theta(x)\|N\| \le (B(x)|N) \le \Theta(x)\|N\|$$

(where $0 < \theta(x) \le \Theta(x)$); it is easy to check by Lemma 1 that

$$-\Theta(x)\|N^-\| + \theta(x)\|N^+\| \le (B(x)|N) \le \Theta(x)\|N^+\| - \theta(x)\|N^-\|. \tag{14}$$

From (12), (13) and (14) we have

$$\begin{aligned}(\theta(x) - a(x)\Lambda)\|N^+\| + (a(x)\lambda - \Theta(x))\|N^-\| &\le (B(x)|N) - a(x)[F(x, M+N) - F(x, M)] \\ &\le (\Theta(x) - a(x)\lambda)\|N^+\| + (a(x)\Lambda - \theta(x))\|N^-\| \\ &\le \sqrt{(\Theta(x) - a(x)\lambda)^2 + (\theta(x) - a(x)\Lambda)^2}\ \|N\|, \end{aligned} \tag{15}$$

by which follows *Conditions* A_x if we take $\delta = 0$ and $a(x) > 0$ *exists* such that

$$\gamma = \sup_{\Omega} \sqrt{(\Theta(x) - a(x)\lambda)^2 + (\theta(x) - a(x)\Lambda)^2} < 1.$$

□

Corollary 1 *Let $n \le 5$ and $\dfrac{n-1}{2\sqrt{n}} \le \dfrac{\lambda}{\Lambda}$. If F is uniformly elliptic, as in Definition 5, then it satisfies Condition A.*

Proof We take, in the proof of Theorem 4, $B(x) = I$, then $\Theta = \sqrt{n}$ and $\theta = 1$. So $\gamma = \sqrt{(\sqrt{n} - a\lambda)^2 + (1 - a\Lambda)^2} < 1$ if $\dfrac{n-1}{2\sqrt{n}} \le \dfrac{\lambda}{\Lambda}$. Moreover $\frac{\lambda}{\Lambda} < 1$, this implies $n \le 5$.

□

Theorem 5 *If F verifies Condition A_x with $B(x) = I$ then F is uniformly elliptic in the sense of Definition 6.*

Proof By (7) it follows for all $M, N \in \mathcal{S}^n N \ge 0$

$$\frac{(1-\delta)(I|N) - \gamma\|N\|}{a(x)} \le F(x, M+N) - F(x, M) \le \frac{\gamma\|N\| + (1+\delta)(I|N)}{a(x)},$$

and hence

$$(I|N)\ \frac{1-\delta-\gamma}{a(x)} \le F(x, M+N) - F(x, M) \le \frac{1+\delta+\gamma}{a(x)}\ (I|N).$$

□

Theorem 6 *Let $F : \Omega\times\mathcal{S}^n \to \mathbf{R}$ be measurable in x and C^1 in the other variables; assume that F is uniformly elliptic, as in Definition 6, with bounds $\lambda(x)$, $\Lambda(x)$ satisfying*

$$\frac{\sqrt{n}-1}{\sqrt{n}+1} < \frac{\lambda(x)}{\Lambda(x)} : \tag{15}$$

then it satisfies $Condition\ A_x$.

Proof We write, by assumption,

$$F(x, M+tN) - F(x, M) = \sum_{i,j=1}^{n} \frac{\partial F(x, M)}{\partial M_{ij}}\, t\, N_{ij} + o(t),\ \text{ as } t \to 0^+. \tag{16}$$

(where $o(t)$ is Landau symbol: $\lim_{t\to 0} \frac{o(t)}{t} = 0$).

From this, by Definition 6, if $t \to 0^+$ we have

$$\lambda(x)\ (I|N) \le \sum_{i,j=1}^{n} \frac{\partial F(x, M)}{\partial M_{ij}} N_{ij} \le \Lambda(x)\,(I|N) \quad \forall M, N \in \mathcal{S}^n,\ \ N \ge 0. \tag{17}$$

Hence

$$\lambda(x)\ \|\xi\|^2 \le \sum_{i,j=1}^{n} \frac{\partial F(x, M)}{\partial M_{ij}} \xi_i\, \xi_j \le \Lambda(x)\, \|\xi\|^2 \quad \forall M \in \mathcal{S}^n,\ \ \forall \xi \in \mathbf{R}^n. \tag{18}$$

Moreover, if $\lambda_1(x, M), \ldots, \lambda_n(x, M)$ are the eigenvalues of the matrix

$$\left\{\frac{\partial F(x, M)}{\partial M_{ij}}\right\}_{i,j=1,\ldots,n},$$

we have that

$$\lambda(x) \le \lambda_i(x, M) \le \Lambda(x), \quad i = 1, \ldots, n,\ \ x \in \Omega,\ \forall M \in \mathcal{S}^n.$$

Its eigenvalues are inclosed in the cube $Q = [\lambda(x), \Lambda(x)]^n$.

The center of Q is $\left(\frac{\lambda(x)+\Lambda(x)}{2}, \ldots, \frac{\lambda(x)+\Lambda(x)}{2}\right)$. We can find a homothety that carries Q in the ball with center $(1, \ldots, 1)$ and radius 1 by taking $a(x) = \frac{2}{\lambda(x)+\Lambda(x)}$, indeed

$$\begin{aligned}\left|(I|N) - a(x)\sum_{i,j=1}^{n} \frac{\partial F(x, M)}{\partial M_{ij}} N_{ij}\right| &\le \|(1, \ldots, 1) - a(x)(\lambda_1(x, M), \ldots, \lambda_n(x, M))\|_{\mathbf{R}^n}\, \|N\| \\ &\le \sqrt{\sum_{i=1}^{n}(1 - a(x)\,\lambda_i(x, M))^2}\, \|N\| \\ &\le \frac{\Lambda(x) - \lambda(x)}{\Lambda(x) + \lambda(x)}\ \sqrt{n}\, \|N\|. \end{aligned} \tag{19}$$

by this it follows $\gamma = \sup_{x\in\Omega} \frac{\Lambda(x)-\lambda(x)}{\Lambda(x)+\lambda(x)}\ \sqrt{n} < 1$ provided that $\inf_{x\in\Omega} \frac{\lambda(x)}{\Lambda(x)} > \frac{\sqrt{n}-1}{\sqrt{n}+1}$.

Finally, by using Lagrange's theorem, we can write by (19)

$$\left|(I|N)-a(x)[F(x,M+N)-F(x,M)]\right|=\left|(I|N)-a(x)\sum_{i,j=1}^{n}\frac{\partial F(x,M_0)}{\partial M_{ij}}N_{ij}\right|\leq\gamma\,\|N\|,$$

where M_0 lies in the segment with end points M, $M+N$. □

Theorem 7 *Let $F:\ \Omega\times\mathcal{S}^n\to\mathbf{R}$ be measurable in x and continuous in the other variables. If F is uniformly elliptic, as in Definition 6, with bounds $\lambda(x)$, $\Lambda(x)$ satisfying*

$$\frac{n-\sqrt{2n-1}}{n-1}\leq\frac{\lambda(x)}{\Lambda(x)}. \tag{20}$$

then it satisfies Condition A_x.

To show this theorem we need the following lemma.

Lemma 2 *F is uniformly elliptic, in the sense of Definition 6 if and only if*

$$-\ \Lambda(x)(I|N^-)+\lambda(x)\,(I|N^+)\leq F(x,M+N)-F(x,M)\leq\Lambda(x)(I|N^+)-\lambda(x)(I|N^-),\quad\forall M,N\in\mathcal{S}^n,\ \forall x\in\Omega. \tag{21}$$

Proof Observe that, by Definition 6:

$$\lambda(x)\,(I|N^+)\leq F(x,M+N^+)-F(M)\leq\Lambda(x)(I|N^+)$$
$$\lambda(x)\,(I|N^-)\leq F(x,M+N^-)-F(M)\leq\Lambda(x)(I|N^-).$$

From this, by taking $\mathcal{M}=M-N^-$, we have

$$\begin{aligned}F(M+N)-F(M)&=F(M+N^+-N^-)-F(M)\\&=F(\mathcal{M}+N^+)-F(\mathcal{M})+F(\mathcal{M})-F(M)\\&\leq\Lambda(x)(I|N^+)+F(\mathcal{M})-F(\mathcal{M}+N^-)\\&\leq\Lambda(x)(I|N^+)-\lambda(x)\,(I|N^-),\end{aligned}$$

and also

$$\begin{aligned}F(M+N)-F(M)&=F(M+N^+-N^-)-F(M)\\&=F(\mathcal{M}+N^+)-F(\mathcal{M})+F(\mathcal{M})-F(M)\\&\geq\lambda(x)(I|N^+)+F(\mathcal{M})-F(\mathcal{M}+N^-)\\&\geq\lambda(x)(I|N^+)-\Lambda(x)\,(I|N^-).\end{aligned}$$

□

Proof (of Theorem 7).
We take $a(x)>0$ and by (21) we can write

$$\begin{aligned}&-\ \Lambda(x)\,a(x)(I|N^-)+\lambda(x)\,a(x)\,(I|N^+)-(I|N^+)+(I|N^-)\\&\leq a(x)[F(x,M+N)-F(x,M)]-(I|N)\\&\leq a(x)\,\Lambda(x)(I|N^+)-a(x)\lambda(x)(I|N^-)-(I|N^+)+(I|N^-),\\&\quad\forall M,N\in\mathcal{S}^n,\quad\forall x\in\Omega,\end{aligned}$$

and then

$$\begin{aligned}
&\left|(I|N) - a(x)[F(x, M+N) - F(x, M)]\right| \\
&\quad \le \max\left\{\left|[a(x)\Lambda(x) - 1](I|N^+) + [1 - a(x)\lambda(x)](I|N^-)\right|,\right. \\
&\qquad \left.\left|[a(x)\Lambda(x) - 1](I|N^-) + [a(x)\lambda(x) - 1](I|N^+)\right|\right\} \\
&\quad \le \sqrt{[a(x)\Lambda(x) - 1]^2 + [a(x)\lambda(x) - 1]^2}\sqrt{(I|N^+)^2 + (I|N^-)^2} \\
&\quad \le \sqrt{[a(x)\Lambda(x) - 1]^2 + [a(x)\lambda(x) - 1]^2}\,\sqrt{n}\,\|N\|M, \quad N \in \mathcal{S}^n, \quad \forall x \in \Omega.
\end{aligned}$$

We obtain the thesis by observing that

$$\sqrt{[a(x)\Lambda(x) - 1]^2 + [a(x)\lambda(x) - 1]^2}\,\sqrt{n} < 1 \quad \text{when } a(x) = \frac{[\lambda(x) + \Lambda(x)]^2}{\lambda(x)^2 + \Lambda(x)^2}$$

provided that

$$\frac{n - \sqrt{2n-1}}{n-1} \le \frac{\lambda(x)}{\Lambda(x)}.$$

□

Remark 2 As we saw in Sect. 1, *Condition A* implies that the Dirichlet Problem (2) has a unique solution; moreover such Condition allows to develop a regularity theory of the solutions in Sobolev spaces (see the references in [4]). Now we ask whether it is possible doing the same by replacing *Condition A* with *Condition* A_x. More precisely:

1. Which hypotheses on $B(x)$, a, γ, δ, allow to use the *near operators theory* in order to show that the Dirichlet problem for an operator F verifying *Condition* A_x is well posed?
2. Which hypotheses on $B(x)$, a, γ, δ, allow to use the techniques of Campanato (see the papers of Campanato in the References) in order to get Schauder inequalities for the solution of the Dirichlet problem for an operator F verifying *Condition* A_x*?*
3. Taking into account the proved connection (see Theorem 4, Theorem 6 and Theorem 7) between uniform ellipticity and *Condition* A_x, is it possible to obtain by *near operators theory* and Campanato regularity techniques the regularity results of [1,9] for the solutions relative to operators F verifying Definition 5 or Definition 6? Is it possible obtain, for the same operators, existence results?

We try to give some partial answers to these questions.
For the first one, we answer by following theorem.

Theorem 8 *Let* $F : \Omega \times \mathcal{S}^n \to \mathbf{R}$ *be measurable in x, continuous in the other variables, satisfying Condition* A_x *with*

$$\begin{aligned}
&a \in L^\infty(\Omega), \quad F(x, 0) \in L^2(\Omega)\|u\|_{H^2(\Omega)} \le \|Bu\|_{L^2(\Omega)} \le C\,\|u\|_{H^2(\Omega)}, \\
&\quad (\text{where } \; Bu = (B(x)|D^2u))\forall u \in H^{2,2}(\Omega),
\end{aligned} \tag{22}$$

If B is a bijection between $H^{2,2} \cap H^{1,2}(\Omega)$ *and* $L^2(\Omega)$ *then Problem (2) is well posed.*

Proof We assume $F(x, 0) = 0$, a.e. in Ω (otherwise we consider $\mathcal{F}(x, M) = F(x, M) - F(x, 0)$) and set

$$A(u) = a(x)\; F(x, D^2u) \quad \mathcal{X} = H^{2,2} \cap H_0^{1,2}(\Omega), \quad \mathcal{B} = L^2(\Omega).$$

Moreover we observe that, by hypothesis (22), $A(u) \in L^2(\Omega)$. Indeed, by *Condition* A_x, we can write

$$\int_\Omega \left|a(x)\ F(x, D^2 u(x))\right|^2\ dx \le 2 \int_\Omega \left\{\left|(B(x)|D^2u(x)) - a(x)\, F(x, D^2u(x))\right|^2 + \left|(B(x)|D^2u(x))\right|^2\right\}\ dx$$
$$\le 2\,[(\gamma + \delta)^2 + 1] \int_\Omega \left|(B(x)|D^2u(x))\right|^2\ dx.$$

At the last, we can end like in the proof of Theorem 2.[2] □

Following theorems give some partial answer to the third question of Remark 2.

Theorem 9 *Let Ω be an open convex set of $\mathbf{R}^n$, and let $F : \Omega \times \mathcal{S}^n \to \mathbf{R}$ be measurable in x, continuous in the other variables, uniformly elliptic as in Definition 5. Moreover $F(x, 0) \in L^2(\Omega)$. Then, if $n \le 5$ and $e\frac{n-1}{2\sqrt{n}} \le \frac{\lambda}{\Lambda}$, Problem 2 is well posed.*

Proof We set

$$Bu = \Delta u, \quad A(u) = F(x, D^2u) \quad \mathcal{X} = H^{2,2} \cap H_0^{1,2}(\Omega), \quad \mathcal{B} = L^2(\Omega)$$

and, using Corollary 1, we may repeat the proof of Theorem 2. □

Theorem 10 *Let Ω be an open convex set of $\mathbf{R}^n$. Let $F : \Omega \times \mathcal{S}^n \to \mathbf{R}$ be measurable in x, continuous in the other variables, uniformly elliptic as in Definition 6. Moreover $F(x, 0) \in L^2(\Omega)$. If*

$$\frac{n - \sqrt{2n-1}}{n-1} \le \frac{\lambda(x)}{\Lambda(x)}, \tag{23}$$

$$0 < \lambda \le \lambda(x) \le \Lambda(x) \le \Lambda, \tag{24}$$

then Problem (2) is well posed.

Proof By Theorem 7, F satisfies Condition A_x with $a(x) = \frac{[\lambda(x)+\Lambda(x)]^2}{\lambda^2(x)+\Lambda^2(x)}$ (see the proof of Theorem). Then, by hypothesis (24), $a \in L^\infty$, so if we set

$$Bu = \Delta u, \quad A(u) = a(x)\ F(x, D^2u) \quad \mathcal{X} = H^{2,2} \cap H_0^{1,2}(\Omega), \quad \mathcal{B} = L^2(\Omega),$$

we have that $A(u) \in L^2(\Omega)$ (see the proof of Theorem 8) and we can end like in the proof of Theorem 2. □

Theorem 11 *Let Ω be an open convex set of $\mathbf{R}^n$, and let $F : \Omega \times \mathcal{S}^n \to \mathbf{R}$ be measurable in x, C^1 in the other variables, uniformly elliptic as in Definition 6. Moreover $F(x, 0) \in L^2(\Omega)$. If*

$$\frac{\sqrt{n}-1}{\sqrt{n}+1} < \frac{\lambda(x)}{\Lambda(x)} \tag{25}$$

$$\lambda \le \lambda(x) \le \Lambda(x) \le \Lambda, \tag{26}$$

then Problem (2) is well posed.

[2] Observe that: $u \in H^{2,2} \cap H_0^{1,2}(\Omega)$ is a solution of the equation $F(x, D^2u(x)) = f(x)$, $f \in L^2(\Omega)$ iff it is a solution of the equation $\alpha(x)\, F(x.D^2u(x)) = g(x)$, where $g(x) = \alpha(x)\, f(x)$, $g \in L^2(\Omega)$.

Proof By Theorem 6, F satisfies *Condition* A_x with $\alpha(x) = \frac{2}{\lambda(x)+\Lambda(x)}$ (see the proof of the Theorem). Then, by hypothesis (26), $a \in L^\infty$, so if we set

$$Bu = \Delta u, \quad A(u) = a(x)\ F(x, D^2u) \quad \mathcal{X} = H^{2,2} \cap H_0^{1,2}(\Omega), \quad \mathcal{B} = L^2(\Omega),$$

we have that $A(u) \in L^2(\Omega)$ (see the proof of Theorem 8) and we can end like in the proof of Theorem 2. □

It seems to us, by these remarks, that the *near operators theory* may be considered another form of the method of continuity. Surely, as it can be seen by the results of the next section, it is a more subtle method.

3 A short survey on near operators theory

The following theorem collects some properties that are preserved by *nearness* among operators.

Theorem 12 *Let $\mathcal{X}$ be a set, $\mathcal{B}$ a Banach space with the norm $\|\cdot\|$, A, B operators from $\mathcal{X}$ to $\mathcal{B}$. If A is near B then:*

(i) if B injective (surjective) then A is injective (surjective) (see S. Campanato [2])*;*
(ii) if $B(\mathcal{X})$ is open in $\mathcal{B}$ then $A(\mathcal{X})$ is open in $\mathcal{B}$ (see A. Tarsia [6])*;*
(iii) if $B(\mathcal{X})$ is dense in $\mathcal{B}$ then $A(\mathcal{X})$ is dense in $\mathcal{B}$ (see A. Tarsia [6])*;*
(iv) if $B(\mathcal{X})$ is compact in $\mathcal{B}$ then $A(\mathcal{X})$ is compact in $\mathcal{B}$ (see A. Tarsia [5]).

Remark 3 Generally the nearness condition is not transitive. But the above properties are in fact transitive. Indeed let $C : \mathcal{X} \longrightarrow \mathcal{B}$ be such that B is near C and A is near B; it is obvious that:

(i) if C is injective (surjective) then A is injective (surjective);
(ii) if $C(\mathcal{X})$ is open in $\mathcal{B}$ then $A(\mathcal{X})$ is open in $\mathcal{B}$;
(iii) if $C(\mathcal{X})$ is dense in $\mathcal{B}$ then $A(\mathcal{X})$ is dense in $\mathcal{B}$;
(iv) if $C(\mathcal{X})$ is compact in $\mathcal{B}$ then $A(\mathcal{X})$ is compact in $\mathcal{B}$.

Moreover we have the following theorem (see [8])

Theorem 13 *Let X and Y be Banach spaces, $\Omega \subset X$ be an open set, and let $A : \Omega \to Y$, $A \in C^1(\Omega)$, $x_0 \in \Omega$. If the Fréchet differential $dA(x_0)$ is a bijection between X and Y, then a constant $\sigma > 0$ exists such that the restriction of A to the ball $S(x_0, \sigma)$ is near $dA(x_0)$.*

It is also possible to show an implicit function theorem (see [8]) where the hypothesis of differentiability is replaced by *nearness.*

The idea of nearness between operators introduced by Campanato allows to define a topology on the set of operators as it follows.

Let us denote with $\mathcal{A}$ and $\mathcal{H}$ the sets:

$$\mathcal{A} = \{B,\ B:\ \mathcal{X} \to \mathcal{B}\}$$
$$\mathcal{H} = \{\Phi, \Phi : \mathcal{X} \to \mathcal{B} \text{ is a bijection between } \mathcal{X} \text{ and } \mathcal{B}\}.$$

We define a topology τ on the set $\mathcal{A}$ such that $\mathcal{H}$ is open in $\mathcal{A}$ with the topology τ. The topology τ can be identified by selecting a neighbourhoods base on $\mathcal{A}$, defined in the following way:

$$\mathcal{U}(B) = \{U_k(B), k \in (0, 1)\}$$

where for each $B \in \mathcal{A}$ and $k \in (0, 1)$ we set

$$U_k(B) = \{A : \mathcal{X} \to \mathcal{B} \text{ such that } \forall x_1,\ x_2 \in \mathcal{X} \text{ we have}$$
$$\|B(x_1) - B(x_2) - [A(x_1) - A(x_2)]\|_{\mathcal{B}} \ \leq\ k\ \|B(x_1) - B(x_2)\|_{\mathcal{B}}\}$$

References

1. Caffarelli, L.A., Cabré, X.: Fully Nonlinear Elliptic Equations, American Mathematical Society Colloquium Publications 43, American Mathematical Society, Providence, RI (1995)
2. Campanato, S.: Recent Results in the Theory of Near Operators, Potential Theory and Degenerate Partial Differential Operators (Parma, 1994), 469–473, Kluwer Acad. Publ., Dordrecht (1995)
3. Campanato, S.: A Cordes type condition for nonlinear nonvariational systems. Rend. Accad. Naz. Sci. XL Mem. Mat. (5) **13**(1), 307–321 (1989)
4. Maugeri, A., Palagachev, D.K., Softova, L.G.: Elliptic and Parabolic Equations with Discontinuous Coefficients, Mathematical Research vol. 9, Wiley-VCH (2000)
5. Tarsia, A.: Recent Developments of the Campanato Theory of Near Operators, Matematiche (Catania), vol. LV, Supplemento no. 2, 197–208 (2000)
6. Tarsia, A.: Some topological properties preserved by nearness among operators and applications to PDE. Czechoslovak Math. J. **46**, 115–133 (1996)
7. Tarsia, A.: On Cordes and Campanato Conditions. Arch. Inequal. Appl. **2**(1), 25–39 (2004)
8. Tarsia, A.: Differential equations and implicit function: a generalization of the near operators theorem. Tolopol Methods Nonlinear Anal. **11**, 115–133 (1998)
9. Trudinger, N.S.: Fully nonlinear, uniformly elliptic equations under natural structure conditions. Trans. Amer. Math. Soc. **278**(2), 751–769 (1983)